

One-way deficit and quantum phase transitions in XX Model

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Quantum correlations including entanglement and quantum discord has drawn much attention in characterizing quantum phase transitions. Quantum deficit originates in questions regarding work extraction from quantum systems coupled to a heat bath [Phys. Rev. Lett. **89**, 180402 (2002)]. It links quantum thermodynamics with quantum correlations and provides a new standpoint for understanding quantum non-locality. In this paper, we evaluate the one-way deficit of two adjacent spins in the bulk for the XX model. In the thermodynamic limit, the XX model undergoes a first order transition from fully polarized to a critical phase with quasi-long-range order with decrease of quantum parameter. We find that the one-way deficit becomes nonzero after the critical point. Therefore, the one-way deficit characterizes the quantum phase transition in the XX model.

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I. INTRODUCTION

The recent development in quantum information theory [1] has provided much insight into quantum phase transitions [2]. Especially, using the quantum correlations to research quantum phase transitions has attracted much attention and has been successful in characterizing a large number of critical phenomena of great interest. In this family, entanglement was the first and most outstanding member to detect various of quantum phase transitions, see [3–7]. Quantum discord is a measure of the difference between the mutual information and maximum classical mutual information, is also used to study quantum phase transitions [8, 9]. Another indication of quantumness that is also found its applications for probing quantum phases and quantum phase transition [10–13].

Besides entanglement and quantum discord, quantum deficit [14–16] originates on asking how to use nonlocal operation to extract work from a correlated system coupled to a heat bath [14]. Oppenheim *et al.* defined the work deficit [14] is a measure of the difference between the information of the whole system and the localizable information [17, 18]. Recently, by means of relative entropy, Streltsov *et al.* [19, 20] give the definition of the one-way information deficit which is also called one-way deficit, which uncovers an important role of quantum deficit as a resource for the distribution of entanglement.

In this paper, we will endeavor to calculate the one-way deficit of two adjacent spins in the bulk of the XX model. In the thermodynamic limit, it undergoes a first order transition from fully polarized to a critical phase with quasi-long-range order with decrease of quantum parameter λ . We find that the one-way deficit becomes nonzero after the critical point. Therefore, the one-way deficit characterizes the quantum phase transition in the XX model. That is, we can employ the deficit to detect

the quantum phase transition point in the XX model.

II. ONE-WAY DEFICIT IN XX MODEL FOR $0 < \lambda < 1$

One-way deficit by von Neumann measurement on one side is given by [21]

$$\Delta^{\rightarrow}(\rho^{ab}) = \min_{\{\Pi_k\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k) - S(\rho^{ab}). \quad (1)$$

We investigate the Spin- $\frac{1}{2}$ XX model with the Hamiltonian as

$$H_{xx} = -\frac{1}{2} \sum_{i=1}^N [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y] - \lambda \sum_{i=1}^N \sigma_i^z, \quad (2)$$

where the interaction constant is taken as the energy unit and λ represents the strength of the external magnetic field and $\sigma^x, \sigma^y, \sigma^z$ are the usual Pauli matrices. Its quantum phase transition is like that in one phase the system is gapped while in the whole region of the other phase the system is critical. The open boundary conditions are assumed, i.e. $N+1 \equiv 0$. The phase diagram is symmetric in respect of λ [22, 23], therefore we only consider the case of positive λ . For $\lambda > 1$ the ground state is polarized. At $\lambda = 1$ the system undergoes a first order quantum transition. In the region $\lambda < 1$ the system is critical. This model can be solved analytically [23] using the following Jordan-Wigner and Fourier transformations

$$d_k = \sqrt{\frac{2}{N+1}} \sum_{l=1}^N \sin\left(\frac{\pi kl}{N+1}\right) \prod_{m=1}^{l-1} \sigma_m^z \sigma_l^-, \quad (3)$$

which turn the Hamiltonian into the diagonalized form in terms of fermion operator as $H_{xx} = \sum_{k=1}^N \Lambda_k d_k^\dagger d_k + N\lambda$ with $\Lambda_k = 2 \left[\cos\left(\frac{\pi k}{N+1}\right) - \lambda \right]$.

In the phase where $0 \leq \lambda < 1$, the ground state corresponds to that having fermions occupied at negative Λ_k

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only. The reduced density matrix for two spins at l and $l+1$ has been obtained in [23] as

$$\rho_{l,l+1} = a_+ |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + a_- |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + b_+ |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + b_- |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + e(|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow|) \quad (4)$$

with

$$\begin{aligned} a_{\pm} &= \frac{1}{4}[1 \pm \langle\sigma_l^z\rangle \pm \langle\sigma_{l+1}^z\rangle + \langle\sigma_l^z\sigma_{l+1}^z\rangle], \\ b_{\pm} &= \frac{1}{4}[1 \pm \langle\sigma_l^z\rangle \mp \langle\sigma_{l+1}^z\rangle - \langle\sigma_l^z\sigma_{l+1}^z\rangle], \\ e &= \frac{1}{2}\langle\sigma_l^x\sigma_{l+1}^x\rangle. \end{aligned} \quad (5)$$

In the thermodynamic limit it can be shown that for bulk spins we have[24]

$$\begin{aligned} \langle\sigma_l^z\sigma_{l+1}^z\rangle &= \left(1 - \frac{2\arccos(\lambda)}{\pi}\right)^2 - \frac{4}{\pi^2}(1 - \lambda^2), \\ \langle\sigma_l^x\sigma_{l+1}^x\rangle &= -\frac{2}{\pi}\sin(\arccos(\lambda)), \\ \langle\sigma_l^z\rangle &= \langle\sigma_{l+1}^z\rangle = 1 - \frac{2\arccos(\lambda)}{\pi}. \end{aligned} \quad (6)$$

Let

$$\begin{aligned} c &= c_1 = c_2 = \langle\sigma_l^x\sigma_{l+1}^x\rangle, \\ c_3 &= \langle\sigma_l^z\sigma_{l+1}^z\rangle, \\ r &= s = \langle\sigma_l^z\rangle. \end{aligned} \quad (7)$$

The density matrix in Eq. (4) is rewritten as

$$\begin{aligned} \rho_{l,l+1} &= \frac{1}{4} \begin{pmatrix} 1+2r+c_3 & 0 & 0 & 0 \\ 0 & 1-c_3 & 2c & 0 \\ 0 & 2c & 1-c_3 & 0 \\ 0 & 0 & 0 & 1-2r+c_3 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 1+r+s+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1+r-s-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-r+s-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1-r-s+c_3 \end{pmatrix}. \end{aligned} \quad (8)$$

The state in Eq. (8) has the following form

$$\rho^{ab} = \frac{1}{4}(I \otimes I + r\sigma_3 \otimes I + I \otimes s\sigma_3 + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i), \quad (9)$$

where $\sigma_1, \sigma_2, \sigma_3$ is Pauli matrix.

The eigenvalues of the X states in Eq. (8) are given by

$$\begin{aligned} u_{\pm} &= \frac{1}{4}(1 - c_3 \pm 2|c|), \\ v_{\pm} &= \frac{1}{4}(1 + c_3 \pm 2|r|). \end{aligned}$$

The entropy is given by

$$\begin{aligned} S(\rho) &= 2 - \frac{1}{4}[(1 - c_3 + 2|c|)\log(1 - c_3 + 2|c|) \\ &\quad + (1 - c_3 - 2|c|)\log(1 - c_3 - 2|c|) \\ &\quad + (1 + c_3 + 2|r|)\log(1 + c_3 + 2|r|) \\ &\quad + (1 + c_3 - 2|r|)\log(1 + c_3 - 2|r|)]. \end{aligned} \quad (10)$$

Next, we evaluate the one-way deficit of the X states in Eq. (8). Let $\{\Pi_k = |k\rangle\langle k|, k = 0, 1\}$ be the local measurement for the party b along the computational base $|k\rangle$; then any von Neumann measurement for the party b can be written as

$$\{B_k = V\Pi_k V^\dagger : k = 0, 1\} \quad (11)$$

for some unitary $V \in U(2)$. For any unitary V ,

$$V = tI + i\vec{y} \cdot \vec{\sigma} = \begin{pmatrix} t + y_3i & y_2 + y_1i \\ -y_2 + y_1i & t - y_3i \end{pmatrix}. \quad (12)$$

with $t \in \mathbb{R}$, $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$, and

$$t^2 + y_1^2 + y_2^2 + y_3^2 = 1, \quad (13)$$

after the measurement B_k , the state ρ^{ab} will be changed into the ensemble $\{\rho_k, p_k\}$ with

$$\rho_k := \frac{1}{p_k}(I \otimes B_k)\rho(I \otimes B_k), \quad p_k = \text{tr}(I \otimes B_k)\rho(I \otimes B_k) \quad (14)$$

To evaluate ρ_k and p_k , we write

$$\begin{aligned} p_k \rho_k &= (I \otimes B_k) \rho (I \otimes B_k) \\ &= \frac{1}{4} (I \otimes V) (I \otimes \Pi_k) [I + r \sigma_3 \otimes I + s I \otimes V^\dagger \sigma_3 V^\dagger \\ &\quad + \sum_{j=1}^3 c_j \sigma_j \otimes (V^\dagger \sigma_j V)] (I \otimes \Pi_k) (I \otimes V^\dagger). \end{aligned}$$

By the relations [25]

$$\begin{aligned} V^\dagger \sigma_1 V &= (t^2 + y_1^2 - y_2^2 - y_3^2) \sigma_1 + 2(ty_3 + y_1 y_2) \sigma_2 \\ &\quad + 2(-ty_2 + y_1 y_3) \sigma_3, \\ V^\dagger \sigma_2 V &= 2(-ty_3 + y_1 y_2) \sigma_1 + (t^2 + y_2^2 - y_1^2 - y_3^2) \sigma_2 \\ &\quad + 2(ty_1 + y_2 y_3) \sigma_3, \\ V^\dagger \sigma_3 V &= 2(ty_2 + y_1 y_3) \sigma_1 + 2(-ty_1 + y_2 y_3) \sigma_2 \\ &\quad + (t^2 + y_3^2 - y_1^2 - y_2^2) \sigma_3, \end{aligned}$$

and

$$\Pi_0 \sigma_3 \Pi_0 = \Pi_0, \Pi_1 \sigma_3 \Pi_1 = -\Pi_1, \Pi_j \sigma_k \Pi_j = 0, \quad (15)$$

for $j = 0, 1, k = 1, 2$, we obtain

$$\begin{aligned} p_0 \rho_0 &= \frac{1}{4} [I + rz_3 I + cz_1 \sigma_1 + cz_2 \sigma_2 + (r + c_3 z_3) \sigma_3] \\ &\quad \otimes (V \Pi_0 V^\dagger), \\ p_1 \rho_1 &= \frac{1}{4} [I - rz_3 I - cz_1 \sigma_1 - cz_2 \sigma_2 + (r - c_3 z_3) \sigma_3] \\ &\quad \otimes (V \Pi_1 V^\dagger), \end{aligned}$$

where

$$\begin{aligned} z_1 &= 2(-ty_2 + y_1 y_3), \\ z_2 &= 2(ty_1 + y_2 y_3), \\ z_3 &= t^2 + y_3^2 - y_1^2 - y_2^2. \end{aligned}$$

Then, we will evaluate the eigenvalues of $\sum_k \Pi_k \rho^{ab} \Pi_k$ by

$$\sum_k \Pi_k \rho^{ab} \Pi_k = p_0 \rho_0 + p_1 \rho_1, \quad (16)$$

and

$$\begin{aligned} &p_0 \rho_0 + p_1 \rho_1 \\ &= \frac{1}{4} (I + r \sigma_3) \otimes I \\ &\quad + \frac{1}{4} (rz_3 I + cz_1 \sigma_1 + cz_2 \sigma_2 + c_3 z_3 \sigma_3) \otimes V \sigma_3 V^\dagger. \end{aligned}$$

The eigenvalues of $p_0 \rho_0 + p_1 \rho_1$ are the same with the eigenvalues of the states $(I \otimes V^\dagger)(p_0 \rho_0 + p_1 \rho_1)(I \otimes V)$, and

$$\begin{aligned} &(I \otimes V^\dagger)(p_0 \rho_0 + p_1 \rho_1)(I \otimes V) \\ &= \frac{1}{4} (I + r \sigma_3) \otimes I \\ &\quad + \frac{1}{4} (rz_3 I + cz_1 \sigma_1 + cz_2 \sigma_2 + c_3 z_3 \sigma_3) \otimes \sigma_3. \quad (17) \end{aligned}$$

By using $z_1^2 + z_2^2 + z_3^2 = 1$, the eigenvalues of the states in the equation (17) are

$$\begin{aligned} \omega_{1,2} &= \frac{1}{4} \left(1 - rz_3 \pm \sqrt{r^2 - 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right), \\ \omega_{3,4} &= \frac{1}{4} \left(1 + rz_3 \pm \sqrt{r^2 + 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right). \end{aligned}$$

The entropy of $\sum_k \Pi_k \rho^{ab} \Pi_k$ is $S(\sum_k \Pi_k \rho^{ab} \Pi_k) = -\sum_{i=1}^4 \omega_i \log \omega_i$. When λ is fixed, r, c, c_3 is constant. It converts the problem about $\min_{\{\Pi_k\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k)$ to the problem about the function of one variable z_3 for minimum. That is

$$\min_{\{\Pi_k\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k) = \min_{\{z_3\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k). \quad (18)$$

By Eqs. (1), (10), (18), the one-way deficit of the X states in Eq. (8) is given by

$$\begin{aligned} &\Delta^\rightarrow(\rho^{ab}) \\ &= \min_{\{\Pi_k\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k) - S(\rho^{ab}) \\ &= \min_{z_3} \left\{ -\frac{1}{4} \left[\left(1 - rz_3 + \sqrt{r^2 - 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \right. \right. \\ &\quad \cdot \log \left(1 - rz_3 + \sqrt{r^2 - 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \\ &\quad + \left(1 - rz_3 - \sqrt{r^2 - 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \\ &\quad \cdot \log \left(1 - rz_3 - \sqrt{r^2 - 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \\ &\quad + \left(1 + rz_3 + \sqrt{r^2 + 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \\ &\quad \cdot \log \left(1 + rz_3 + \sqrt{r^2 + 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \\ &\quad + \left(1 + rz_3 - \sqrt{r^2 + 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \\ &\quad \cdot \log \left(1 + rz_3 - \sqrt{r^2 + 2rc_3 z_3 + c^2 + (c_3^2 - c^2) z_3^2} \right) \left. \right\} \\ &\quad + \frac{1}{4} [(1 - c_3 + 2|c|) \log(1 - c_3 + 2|c|) \\ &\quad + (1 - c_3 - 2|c|) \log(1 - c_3 - 2|c|) \\ &\quad + (1 + c_3 + 2|r|) \log(1 + c_3 + 2|r|) \\ &\quad + (1 + c_3 - 2|r|) \log(1 + c_3 - 2|r|)]. \quad (19) \end{aligned}$$

Note that $\Delta^\rightarrow(\rho^{ab})$ is an even function for the variable z_3 , so we can focus on $z_3 \in [0, 1]$ instead of $[-1, 1]$.

For example, we set $\lambda = 0.6$, then $r = 0.409666, c = -0.509296, c_3 = -0.0915564$, and obtain that the value of the one-way deficit is 0.418314.

III. ONE-WAY DEFICIT IN XX MODEL FOR $\lambda > 1$

When $\lambda > 1$, Λ_k is negative for all k . Therefore the ground state has N fermions, which is translated as all spins up in spin language. It is straightforward to see that in this case the one-way deficit of the ground state or that of any part of its reduced density operator is always zero.

In fact, for $\lambda > 1$ the ground state is polarized, whose reduced density matrix for two spins at l and $l+1$ has turned into

$$\rho_{l,l+1} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| \quad (20)$$

The density matrix in Eq. (20) is rewritten as

$$\rho_{l,l+1} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \quad (21)$$

Next, we evaluate the one-way deficit of the states in Eq. (21). Let $\{\Pi_k = |k\rangle\langle k|, k = 0, 1\}$ be the local measurement for the party b along the computational base $|k\rangle$. By (11), (12), (14), after the measurement B_k , the state in Eq. (21) will be changed into the ensemble $\{\bar{\rho}_k, \bar{p}_k\}$. After some algebraic calculus, we obtain

$$\begin{aligned} \bar{p}_0 \bar{\rho}_0 &= (t^2 + y_3^2) |0\rangle\langle 0| \otimes V |0\rangle\langle 0| V^\dagger, \\ \bar{p}_1 \bar{\rho}_1 &= (y_1^2 + y_2^2) |0\rangle\langle 0| \otimes V |1\rangle\langle 1| V^\dagger \end{aligned}$$

Then, we will evaluate the eigenvalues of $\sum_k \Pi_k \rho_{l,l+1} \Pi_k$ by

$$\sum_k \Pi_k \rho_{l,l+1} \Pi_k = \bar{p}_0 \bar{\rho}_0 + \bar{p}_1 \bar{\rho}_1, \quad (22)$$

and

$$\bar{p}_0 \bar{\rho}_0 + \bar{p}_1 \bar{\rho}_1 = |0\rangle\langle 0| \otimes V \begin{pmatrix} t^2 + y_3^2 & 0 \\ 0 & y_1^2 + y_2^2 \end{pmatrix} V^\dagger. \quad (23)$$

The eigenvalues of $\bar{p}_0 \bar{\rho}_0 + \bar{p}_1 \bar{\rho}_1$ are $t^2 + y_3^2, y_1^2 + y_2^2, 0, 0$. The entropy of $\sum_k \Pi_k \rho_{l,l+1} \Pi_k$ is

$$\begin{aligned} S(\sum_k \Pi_k \rho_{l,l+1} \Pi_k) &= -(t^2 + y_3^2) \log(t^2 + y_3^2) \\ &\quad - (y_1^2 + y_2^2) \log(y_1^2 + y_2^2). \end{aligned} \quad (24)$$

By Eq. (13), we know that $S(\sum_k \Pi_k \rho_{l,l+1} \Pi_k)$ is binary entropy function. When $t = 0, y_3 = 0, y_1^2 + y_2^2 = 1$ or $y_1 = 0, y_2 = 0, t^2 + y_3^2 = 1$, the function $S(\sum_k \Pi_k \rho_{l,l+1} \Pi_k)$ reaches the minimum 0. By the entropy of $\rho_{l,l+1}$ in Eq. (21) being zero and Eq. (1), the one-way deficit of the states in Eq. (21) is 0.

In Fig. 1, we draw the curve of one-way deficit of two adjacent spins in the bulk of XX model in $\lambda \in [0, 1.5]$.

We find that the one-way deficit is nonzero in the domain $\lambda \in [0, 1)$ and then becomes zero when $\lambda \geq 1$. As the XX model undergoes a first order transition at the critical point $\lambda = 1$ from fully polarized to a critical phase with quasi-long-range order, we conclude that one-way deficit can be used to detect quantum phase of the XX model, and moreover, may reveal the insight of phase transition by quantum correlations.

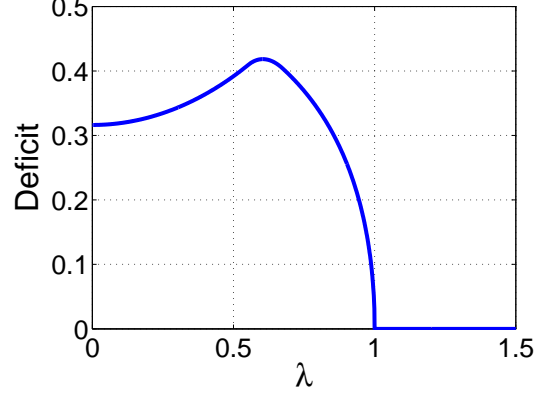


FIG. 1: (Color online) One-way deficit of two adjacent spins in the bulk for the XX model in the thermodynamic limit as a function of the quantum parameter λ .

IV. CONCLUSION

We have given a method to evaluate the one-way deficit two adjacent spins in the bulk for the XX model in the thermodynamic limit. We have drawn the curve of one-way deficit of the XX model. We find that we can use one-way deficit to detect quantum phase of the XX model. We find that the one-way deficit becomes zero when $\lambda \geq 1$. Therefore, the one-way deficit can characterizes the quantum phase transition in the XX model. This may shed lights on the study of properties of quantum correlations in different quantum phases.

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